# RELAXATION APPROXIMATION IN THE THEORY OF SHEAR TURBULENCE

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## Abstract

Leslie's perturbative treatment of the direct interaction approximation for shear turbulence (*Modern Developments in the Theory of Turbulence*, 1972) is applied to derive a time dependent model for the Reynolds stresses. The stresses are decomposed into tensor components which satisfy coupled linear relaxation equations; the present theory therefore differs from phenomenological Reynolds stress closures in which the time derivatives of the stresses are expressed in terms of the stresses themselves. The theory accounts naturally for the time dependence of the Reynolds normal stress ratios in simple shear flow. The distortion of wavenumber space by the mean shear plays a crucial role in this theory.

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## I. Introduction

Time dependent theories of turbulence can be derived by closing correlations in the exact stress rate equations in terms of the stresses themselves.[1,2] The present work is based instead on Leslie's observation[3] that the direct interaction approximation (DIA) equations for shear flow[4] can be solved by a perturbation series in the strain rate; the goal is to apply Leslie's ideas to single point time dependent turbulence modeling. This application of DIA is natural since DIA is a time dependent theory, but since the fully transient time evolution of turbulence cannot be described at the single point level, approximations are required. Following Leslie and Yoshizawa,[5] we assume that the imposed strain is small enough and varies slowly enough to permit treating it as a perturbation of a background state of isotropic turbulence. In this limit, single point relaxation approximations[6] can be derived by requiring that the model agree with the universal long and short time limits of the two point theory.[7]

The model which results has the "viscoelastic" character emphasized by Crow:[8] in simple shear flow in which $\partial U_i/\partial x_j = S\delta_{i1}\delta_{j2}$, the shear stress $\tau$ is given at short times by the rapid distortion theory (RDT) limit

$$\tau = \frac{4}{15}K\ t\ S \tag{1}$$

where $K$ is the turbulence kinetic energy and $t$ is time; at long times $\tau$ is given instead by the eddy viscosity formula

$$\tau = C_\nu \frac{K^2}{\varepsilon}\ S \tag{2}$$

where $\varepsilon$ is the dissipation rate. The short and long time limits Eqs. (1) and (2) are also satisfied by previous time dependent models including the time dependent shear stress model of Hanjalic and Launder[9]

$$\dot{\tau} = -C_R\frac{\varepsilon}{K}\tau + \frac{4}{15}KS \tag{3}$$

by the time dependent eddy viscosity formula of Ref. 6, and by an RDT-based model attributable to Townsend, Hunt, and Maxey.[10]

The present model extends these short and long time constraints to second order effects including the inequality of normal stresses in simple shear flow. In this case, the short time limit was computed by Maxey;[10] the long time limit is the nonlinear eddy viscosity formula of Pope[11] (see also Ref. 12, 13 and 14). An interesting property of the Reynolds normal stresses is the time dependence of their ratios: defining $b_{ij} = 2/3\delta_{ij} - \tau_{ij}/2K$, the short time ratios are[10]

$$b_{11} : b_{22} : b_{33} = 8 : -13 : 5 \tag{4}$$

whereas at long times, very nearly[1,2]

$$b_{11} : b_{22} : b_{33} = 4 : -3 : -1 \tag{5}$$

Previous theories do not give entirely satisfactory accounts of the transition between these short and long time ratios. The complete Launder-Reece-Rodi (LRR) stress transport model[15] which generalizes Eq. (3) predicts that the long and short time normal stress ratios are equal; the model is calibrated to predict the practically more significant long time ratios correctly. More complex second moment closures which are nonlinear in the Reynolds stresses could certainly be made consistent with both limits, but the nonlinearity is itself problematical, especially when rapid distortion limits are relevant.[1,2] The RDT model of Ref. 10 automatically agrees with the short time limit when it is formulated for initially isotropic turbulence but the corresponding long time limits are not satisfactory; the model is therefore recalibrated by assuming initially anisotropic turbulence. The degree of anistropy is chosen to match the long time limits; this adjustment slightly alters the short time ratios. Despite some quantitatively successful predictions of certain time dependent flows using this model,[16] this account of time dependent turbulence does not preclude further investigation.

Unlike the advanced second moment closures described in Ref. 1 and 2, the present model is entirely linear; the complexity required to account for the second order long and short time limits enters through a decomposition of the Reynolds stress into tensor components with distinct time dependent behavior. The decomposition is determined by a geometric feature of shear turbulence, the distortion of wavenumber space by the mean shear. This effect is often treated as a weak mechanism of energy transfer by eddy distortion;[18] it proves to be negligible at short times but sufficiently important at long times to bring about the transition from Eq. (4) to Eq. (5). The introduction of quantities with different time dependence is generally consistent with Weinstock's suggestions[19] that the shear and normal stresses have distinct relaxational behavior.

It must be emphasized that the present theory is limited to small strain rates with rapid distortion theory describing only the short time limit. The more difficult problem of turbulence at high strain rates is left to further investigations.

## II. Simplified DIA analysis of shear turbulence

The DIA generalized Langevin model[20,21] for isotropic turbulence is

$$\frac{\partial}{\partial t}u_i(\mathbf{k},t) + \int_0^t ds\ \eta(k,t,s)u_i(\mathbf{k},s) = f_i(\mathbf{k},t) \tag{6}$$

where $\eta$ is a damping function and $f$ is a random force. In Eq. (6), the viscous term has been dropped. Define the response function $G$ as the inverse operator for the left side of Eq. (6) so that

$$\frac{\partial}{\partial t}G(k,t,s) + \int_s^t dr\ \eta(k,t,r)\ G(k,r,s) = 0 \tag{7}$$

and

$$G(k,t,t) = 1, \quad G(k,t,s) = 0 \text{ for } t < s \tag{8}$$

Define the two time correlation function by

$$Q_{ij}(\mathbf{k},t,s)\delta(\mathbf{k}+\mathbf{k}') \ = < u_i(\mathbf{k},t)u_j(\mathbf{k}',s) > \tag{9}$$

so that in isotropic turbulence

$$Q_{ij}(\mathbf{k},t,s) = Q(k,t,s)P_{ij}(\mathbf{k})$$

where $k^2 = \mathbf{k} \cdot \mathbf{k}$ and $P_{ij}$ is the transverse projection operator

$$P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j k^{-2}$$

Then the damping function $\eta$ is related to $G$ and $Q$ by[20,21]

$$\eta(k,t,s) = \int_{\mathbf{k}=\mathbf{p}+\mathbf{q}} d\mathbf{p} d\mathbf{q}\ P_{tmn}(\mathbf{k}) P_{mst}(\mathbf{p}) P_{ns}(\mathbf{q}) G(p,t,s) Q(q,t,s) \tag{10}$$

where

$$P_{imn}(\mathbf{k}) = k_m P_{in}(\mathbf{k}) + k_n P_{im}(\mathbf{k})$$

The random force $f_i$ in Eq. (6) is defined in terms of independent Gaussian random fields $\xi_i, \xi_i'$ such that

$$< \xi_i(\mathbf{k},t)\xi_j(\mathbf{k}',s) > = < \xi_i'(\mathbf{k},t)\xi_j'(\mathbf{k}',s) > = \ Q_{ij}(\mathbf{k},t,s)\delta(\mathbf{k}+\mathbf{k}') \tag{11}$$

by

$$f_i(\mathbf{k},t) = P_{imn}(\mathbf{k}) \int_{\mathbf{k}=\mathbf{p}+\mathbf{q}} d\mathbf{p}\ d\mathbf{q}\ \xi_m(\mathbf{p},t)\xi_n'(\mathbf{q},t) \tag{12}$$

The DIA equations[20,21] for $G$ and $Q$ follow from Eqs. (6)-(12). The existence of this Langevin model insures the realizability of DIA and also brings forward some analogies to statistical physics, where Eq. (6) arises as a generic model.[22]

The DIA theory of shear turbulence in the special case of spatially constant mean velocity gradient[3,4] can also be formulated as a generalized Langevin model

$$\frac{\partial}{\partial t}\ u_i(\mathbf{k},t) + \int_0^t\ ds\ \eta_{im}(\mathbf{k},t,s)\ u_m(\mathbf{k},s) = \mathrm{S}_{im}(\mathbf{k},t) u_m(\mathbf{k},t) + f_i(\mathbf{k},t) \tag{13}$$

where the mean shear operator[18] is

$$\mathrm{S}_{im}(t) = -\frac{\partial U_i}{\partial x_m}(t) + 2k^{-2} k_i k_p \frac{\partial U_p}{\partial x_m}(t) + \delta_{im} k_s \frac{\partial U_s}{\partial x_r}(t) \frac{\partial}{\partial k_r} \tag{14}$$

and the damping, described by the tensor

$$\eta_{im}(\mathbf{k},t,s) = \int_{\mathbf{k}=\mathbf{p}+\mathbf{q}} d\mathbf{p} d\mathbf{q} P_{ipn}(\mathbf{k}) P_{rsm}(\mathbf{p}) G_{pr}(\mathbf{p},t,s) Q_{ns}(\mathbf{q},t,s) \tag{15}$$

need not be isotropic. The first two terms in the shear operator of Eq. (14) are the production and rapid pressure strain terms; the third is generally described as a relatively weak mechanism of energy transfer due to eddy distortion by the mean shear.[18] This effect can also be considered purely geometrically as a distortion of wavevector space. The random force is defined as above in Eqs. (11),(12); however, the velocity correlation in Eq. (11) must not be assumed isotropic. The shear flow DIA equations[4] for $G_{ij}$ and $Q_{ij}$ follow from Eqs. (13)-(15).

The equations of shear flow DIA seem to defy analysis; even numerical investigation is hampered by the need to resolve a three dimensional anisotropic energy spectrum.[23] However, a useful simplification was suggested by Leslie:[3] introduce an isotropic background field $u^{(0)}$ satisfying Eq.(6), which we rewrite as

$$\frac{\partial}{\partial t} u_i^{(0)}(\mathbf{k},t) + \int_0^t ds\ \eta^{(0)}(k,t,s) u_i^{(0)}(\mathbf{k},s) \ = \ f_i^{(0)}$$

to emphasize that the damping and forcing are isotropic. Formulated as a Langevin equation, Leslie's model is

$$\frac{\partial}{\partial t} u_i(\mathbf{k},t) + \int_0^t ds\ \eta^{(0)}(k,t,s)\ u_i(\mathbf{k},s) = f_i^{(0)}(\mathbf{k},t) + \mathrm{S}_{ip}(\mathbf{k},t) u_p(\mathbf{k},t) \tag{16}$$

in which the damping and forcing are isotropic and independent of the mean shear. Eq. (16) can be rewritten in terms of the isotropic Green's function $G^{(0)}$, the solution of Eqs. (7),(8) as

$$u_i(\mathbf{k},t) = \int_0^t ds\ G^{(0)}(k,t,s)\ [f_i^{(0)}(\mathbf{k},s) + P_{iq}(\mathbf{k})\mathrm{S}_{qp}(\mathbf{k},s) u_p(\mathbf{k},s)] \tag{17}$$

This model ignores any effects of the mean shear on the nonlinear transfer mechanism, modeled by the damping and forcing in the Langevin equation. This is certainly an oversimplification. However, a large scale shear acting against an isotropic transfer mechanism provides a plausible picture of shear turbulence. Moreover, the Kolmogorov time scale $\sim k^{-2/3}$ is much smaller than the time scale of the mean shear at sufficiently small scales. It is reasonable to expect the perturbative treatment to be valid at such scales.

Because of this simplification, it is important to compare Leslie's model with the full DIA shear flow equations. Denote the solution of Eq. (16) by $u_i^{[1]}$. Define the first order correction to the Green's function $G_{ij}^{[1]}$ so that

$$u_i^{[1]}(\mathbf{k},t) \ = \int_0^t ds\ G_{ip}^{[1]}(\mathbf{k},t,s) f_p^{(0)}(\mathbf{k},s) \tag{18}$$

The field $u_i^{[1]}$ can be used to construct a corrected correlation tensor

$$Q^{[1]}(\mathbf{k},t,s)\delta(\mathbf{k}+\mathbf{k}') \ = < u^{[1]}(\mathbf{k},t) u^{[1]}(\mathbf{k}',s) > \tag{19}$$

Substituting the corrected quantities $G_{ij}^{[1]}, Q_{ij}^{[1]}$ in Eq. (15) yields a corrected damping $\eta_{ij}^{[1]}$ and substituting in Eqs.(11),(12) yields a corrected forcing $f_i^{[1]}$. This approximation scheme can be iterated. Define $u_i^{[n]}$ for $n \geq 2$ as the solution of

$$\frac{\partial}{\partial t} u_i^{[n]} + \int_0^t ds\ \eta_{ip}^{[n-1]} u_p^{[n]} = \mathrm{S}_{ip} u_p^{[n]} + f_i^{[n-1]}$$

Define $G_{ij}^{[n]}$ and $Q_{ij}^{[n]}$ given $u_i^{[n]}$ by analogy to Eqs. (18),(19). If the $u_i^{[n]}$ approach a limit, the corresponding limits of $G_{ij}^{[n]}$ and $Q_{ij}^{[n]}$ are a solution of shear flow DIA. Leslie's model is the first step in this iterative construction. It is therefore a rational approximation which can be systematically corrected by further iteration.

### III. Time dependent linear eddy viscosity

To derive the time dependent linear eddy viscosity, we follow Leslie[3] and expand Eq. (17) in powers of the mean strain rate about an isotropic background state $u^{(0)}$:

$$u = u^{(0)} + u^{(1)} + \cdots \tag{20}$$

The random force $f^{(0)}$ which maintains the background field[3] is taken to be of the same order as $u^{(0)}$; therefore, $u^{(1)}$ is given by

$$u_i^{(1)}(\mathbf{k},t) = \int_0^t ds\ G^{(0)}(k,t,s)\ P_{im}(\mathbf{k})\ \mathrm{S}_{mn}\ (\mathbf{k},s)\ u_n^{(0)}\ (\mathbf{k},s) \tag{21}$$

If, corresponding to Eq. (20) the single time correlation function is expanded as

$$Q(k,t,t) = Q^{(0)}(k,t) + Q^{(1)}(k,t) + \cdots \tag{22}$$

where

$$Q_{ij}^{(1)}(\mathbf{k},t)\delta(\mathbf{k}+\mathbf{k}')\ = < u_i^{(1)}\ (\mathbf{k},t)\ u_j^{(0)}(\mathbf{k}',t)\ +\ u_i^{(0)}(\mathbf{k},t)u_j^{(1)}(\mathbf{k}',t)\ >$$

and the higher order correlations are defined similarly, then Eq. (21) implies[3]

$$\begin{aligned} Q_{ij}^{(1)}(\mathbf{k},t) = \int_0^t ds\{\ & G^{(0)}(k,t,s)(-\frac{\partial U_i}{\partial x_r} + 2k_ik_pk^{-2}\frac{\partial U_p}{\partial x_r})Q_{rj}^{(0)}(k,t,s) + (ij) \\ & + G^{(0)}(k,t,s)k_r\ \frac{\partial U_r}{\partial x_n}\ \frac{\partial}{\partial k_n}\ Q_{ij}^{(0)}(k,t,s) \\ & - Q_{ij}^{(0)}(k,t,s)k_r\ \frac{\partial U_r}{\partial x_n}\ \frac{\partial}{\partial k_n}G^{(0)}(k,t,s)\} \end{aligned} \tag{23}$$

where $(ij)$ denotes index interchange in the immediately preceding term. The $k$ derivative terms represent the distortion of wavevector space by the mean shear. A decomposition of the Reynolds stress follows from Eq. (22):

$$\tau_{ij} =\ - < u_iu_j > = \tau_{ij}^{(0)} + \tau_{ij}^{(1)} + \cdots$$

where

$$\tau_{ij}^{(n)}(t)\ =\ -\int d\mathbf{k}\ Q_{ij}^{(n)}(\mathbf{k},t) \tag{24}$$

Note that $\tau^{(n)}$ is homogeneous of degree $n$ in the mean velocity gradient. In applying Leslie's theory, it will be convenient to invoke the Markovianized version of DIA described in Appendix II. Evaluation of the angular integrals in Eqs. (23),(24) using Eq. (70) leads to[3]

$$\tau_{ij}^{(1)}(t) = \int_0^t ds \int_0^\infty dk \left\{ \frac{4}{15}\, G(k,t,s)^2 - \frac{2}{15}\, G(k,t,s) k\, \frac{d}{dk}\, G(k,t,s) \right\} E(k,s)\, S_{ij}(s) \quad (25)$$

where $E(k,s)$ is the energy spectrum at time $s$, and

$$S_{ij} \;=\; \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}$$

is the strain rate. In Eq. (25) and what follows, the superscript (0) on the background field descriptors has been dropped.

An equation similar to Eq. (25) has recently been derived by Woodruff[24] by a scale separation expansion similar to Yoshizawa's and has been developed into a subgrid scale stress model. Our goal is instead the development of a single point model. Eq. (25) suggests a decomposition of the first order stress,

$$\tau_{ij}^{(1)} = \frac{4}{15} T_{ij}^{(0)} + \frac{2}{15} T_{ij}^{(1)} \quad (26)$$

where

$$\begin{aligned} T_{ij}^{(0)}(t) &= \int_0^t ds \int_0^\infty dk\; G(k,t,s)^2\; E(k,s)\; S_{ij}(s) \\ T_{ij}^{(1)}(t) &= -\int_0^t ds \int_0^\infty dk\; G(k,t,s) k\; \left[\frac{d}{dk}\, G(k,t,s)\right]\; E(k,s) S_{ij}(s) \end{aligned} \quad (27)$$

The term $T^{(1)}$ exhibits the distortion of wavevector space by the mean shear. The time dependent model provided by Eqs. (26), (27) is purely formal because the time evolution of $G$ and $E$ is not known but must be found in general by solving DIA to determine the fully transient spectral dynamics. At this level of generality, single point modeling is not possible; extracting a practical single point model requires further restrictions on the time dependence. We can retain some of the time dependence of Eqs. (26), (27) following Smith and Yakhot[7] by requiring that the single point model agree with the essentially universal short and long time limits of Eq. (27). It must be stressed that this model is necessarily of quite restricted applicability; it cannot describe arbitrary complex strain rate histories.

At short times, we have

$$\begin{aligned} E(k,t) &= E(k) + O(t) \\ S_{ij}(t) &= S_{ij} + O(t) \end{aligned}$$

and in view of Eqs. (7), (8),

$$\begin{aligned} G(k,t,s) &= 1 + O(t-s)^2 \\ \frac{dG}{dk}(k,t,s) &= O(t-s)^2 \end{aligned}$$

consequently,

$$T_{ij}^{(0)} = KtS_{ij} + O(t^2), \qquad T_{ij}^{(1)} = O(t^2) \tag{28}$$

Substituting in Eq. (26), there results

$$\tau_{ij}^{(1)} \;=\; \frac{4}{15}\; Kt\; S_{ij} + O(t^2) \tag{29}$$

in agreement with Crow's[8] RDT calculation. The long time limit is a hypothetical steady state defined by setting $E(k,t)$ and $S(t)$ to constants, setting $G$ equal to its time stationary form

$$G(k,t,s) \;=\; G(k,t-s) \;=\; \exp[-\eta(k)(t-s)] \tag{30}$$

from Appendix II, Eq. (68), and setting $t = \infty$ in Eq. (27). Exactly homogeneous shear turbulence is never naturally in a steady state; this long time limit occurs only locally in inhomogeneous flows like the log layer where it can be maintained by strong turbulent diffusion.

After substituting in Eq. (26), the result is the eddy viscosity representation

$$\tau_{ij}^{(1)} = \nu_T S_{ij} \tag{31}$$

where

$$\nu_T = \frac{4}{15}\int_0^\infty dk\;\theta_0 E + \frac{2}{15}\int_0^\infty dk\;k\eta'\theta_1 E(k) \tag{32}$$

and $\theta_i$ are time moments of the stationary Green's function of Eq. (30),

$$\theta_n = \int_0^\infty d\sigma\;\sigma^n G(k,\sigma)^2 \;=\; \frac{n!}{(2\eta)^{n+1}} \tag{33}$$

Results like Eq. (32) which express turbulent transport coefficients in terms of integrals of inertial range quantities also arise in Yoshizawa's theories.[5] In view of the appearance of the time integrals $\theta_n$ of response functions defined in Eq. (33), Eq. (32) can be compared to Kubo formulas[25] expressing molecular transport coefficients in terms of integrals of equilibrium correlation functions.

It remains to construct a relaxation model for $T^{(0)}$ and $T^{(1)}$ which is consistent with these limits. In both limits, the mean shear can be considered constant; therefore, it suffices to consider the time evolution of the quantities

$$t^{(0)} = \int_0^\infty dk\;\int_0^t\;ds\;G(k,t,s)^2\;E(k,s) \tag{34}$$

$$t^{(1)} = \int_0^\infty dk\;\int_0^t\;ds\;k\eta'(k)\;(t-s)\;G(k,t,s)^2\;E(k,s) \tag{35}$$

In Eq. (35), the $k$ derivative of Eq. (27) has been evaluated using the time stationary form for $G$ of Eq. (30). Approximate the time evolution of $t^{(0)}$ by the relaxation model

$$\dot{t}^{(0)} = K - C_R^{(0)}\Theta t^{(0)} \tag{36}$$

where

$$\Theta = \frac{\varepsilon}{K}$$

is a frequency of the largest turbulent scales and the relaxation constant $C_R^{(0)}$ is to be determined. The relaxation model Eq. (36) has been chosen to agree with the short time limit of Eq. (34)

$$t^{(0)} \; = \; Kt + O(t^2) \text{ for } t \to 0$$

To match long time limits, set $\dot{t}^{(0)} = 0$ in Eq. (36) and replace $t^{(0)}$ by its long time limit computed from Eq. (34). The result is

$$C_R^{(0)}\Theta = \frac{\int_0^\infty \; dk \; E}{\int_0^\infty \; dk \; \theta_0 E} \tag{37}$$

This procedure can also be applied to the term $t^{(1)}$ in Eq. (35). The relation

$$\dot{t}^{(1)} = \int_0^\infty \; dk \; \int_0^t \; ds \; \{k\eta'(k) + 2\eta(k)k\eta'(k)(t-s)\} G(k,t,s)^2 E(k,s) \tag{38}$$

suggests the relaxation approximation

$$\dot{t}^{(1)} = C_S^{(1)}\Theta t^{(0)} - C_R^{(1)}\Theta t^{(1)} \tag{39}$$

This relaxation model has the same short time limit, $t^{(1)} = O(t^2)$ as Eq. (35). Since only the $O(t)$ term in Eq. (34) has been matched, it would be inconsistent to match terms of order $t^2$ in Eqs. (35) and (39). Instead, $C_S^{(1)}$ is determined by equating the long time limit of $C_S^{(1)}\Theta t^{(0)}$ to the long time limit of the first term on the right side of Eq. (38):

$$C_S^{(1)}\Theta = \frac{\int_0^\infty dk \; k\eta' \; \theta_0 E}{\int_0^\infty dk \; \theta_0 E} \tag{40}$$

Now $C_R^{(1)}$ can be determined by setting $\dot{t}^{(1)} = 0$ in Eq. (39), replacing $t^{(1)}$ by its long time limit evaluated from Eq. (35). The result is

$$C_R^{(1)}\Theta = \frac{\int_0^\infty dk \; 2\eta \; k\eta' \; \theta_1 E}{\int_0^\infty dk \; k\eta' \; \theta_1 E} \tag{41}$$

Substituting the relaxation approximations Eqs. (36), (39) for $t^{(i)}$ into the definitions of $T^{(i)}$ in Eqs. (26), (27), leads to the relaxation approximation for shear stress

$$\begin{aligned}
\tau_{ij}^{(1)} &= \frac{4}{15}T_{ij}^{(0)} + \frac{2}{15}T_{ij}^{(1)} \\
\dot{T}_{ij}^{(0)} &= KS_{ij} - C_R^{(0)}\Theta T_{ij}^{(0)} \\
\dot{T}_{ij}^{(1)} &= C_S^{(1)}\Theta T_{ij}^{(0)} - C_R^{(1)}\Theta T_{ij}^{(1)}
\end{aligned} \tag{42}$$

Eqs. (37), (40), and (41) express the relaxation constants $C_R^{(0)}, C_R^{(1)}, C_S^{(1)}$ in terms of the energy spectrum $E(k)$ and damping function $\eta(k)$ which can be assumed to have Kolmogorov inertial range forms

$$E(k) = C_K \varepsilon^{2/3} k^{-5/3} \quad \eta(k) = C_T \varepsilon^{1/3} k^{2/3} \quad \text{for} \quad k \geq k_0$$

where $C_K$ and $C_T$ are universal inertial range constants, and $k_0$ is a (nonuniversal) cutoff for inertial range scaling in any particular problem. The dependence of the relaxation constants on the cutoff scale $k_0$ and on the form of the spectrum and relaxation function in the far infrared region $k \leq k_0$ must be investigated. Rewrite Eq. (37), for example, as a formula for $C_R^{(0)}$ using the definition of $\Theta$:

$$C_R^{(0)} = \frac{1}{\varepsilon} \frac{\left[\int_0^\infty dk \ E(k)\right]^2}{\int_0^\infty dk \ \theta_0 E(k)}$$

and substitute the simplest expressions for $E$ and $\eta$ in which $E(k) = \eta(k) = 0$ for $k \leq k_0$. Then both numerator and denominator scale as $k_0^{-4/3}$; $C_R^{(0)}$ is therefore independent of the cutoff scale $k_0$. Consequently, the relaxation constants can be evaluated in the infinite Reynolds number limit in which $k_0 \to 0$ and treated as universal properties of the inertial range alone. Thus,

$$C_R^{(0)} = 6 C_K C_T$$

This result is approximate, but the exact value given by Eq. (37) depends only weakly on $k_0$ and the particular form of the spectrum and time scale when $k \leq k_0$.

## IV. Time dependent nonlinear eddy viscosity

Calculation of Leslie's expansion to second order is tedious but straightforward. The result is

$$\begin{aligned} Q_{ij}^{(2)}(\mathbf{k}, t) = \sum_{1 \leq N \leq 6} I^{(N)} [ & a^{(N)} \frac{\partial U_i}{\partial x_p}(s) \frac{\partial U_j}{\partial x_p}(r) + b^{(N)} \frac{\partial U_i}{\partial x_p}(s) \frac{\partial U_p}{\partial x_j}(r) + c^{(N)} \frac{\partial U_p}{\partial x_i}(s) \frac{\partial U_j}{\partial x_p}(r) \\ & + d^{(N)} \frac{\partial U_p}{\partial x_i}(s) \frac{\partial U_p}{\partial x_j}(r) + e^{(N)} \delta_{ij} \frac{\partial U_p}{\partial x_q}(s) \frac{\partial U_q}{\partial x_p}(r) + f^{(N)} \delta_{ij} \frac{\partial U_p}{\partial x_q}(s) \frac{\partial U_p}{\partial x_q}(r)] \\ & + (ij) \end{aligned} \tag{43}$$

in which the $I^{(N)}$ are integral operators

$$I^{(1)} = \int_0^t ds \ G(k, t, s) \int_0^s dr \ G(k, s, r) G(k, t, r) Q(k, r)$$

$$I^{(2)} = \int_0^t ds \ G(k, t, s) \int_0^s dr \ G(k, s, r) k \frac{d}{dk} [G(k, t, r) Q(k, r)]$$

$$I^{(3)} = \int_0^t ds \ G(k, t, s) \int_0^s dr \ G(k, s, r) k^2 \frac{d^2}{dk^2} [G(k, t, r) Q(k, r)]$$

$$I^{(4)} = \int_0^t ds\ G(k,t,s) \int_0^t dr\ G(k,t,r) G(k, \mid s,r \mid) Q(k,r) \tag{44}$$
$$I^{(5)} = \int_0^t ds\ G(k,t,s) \int_0^t dr\ G(k,t,r) k \frac{d}{dk} [G(k, \mid s,r \mid) Q(k,r)]$$
$$I^{(6)} = \int_0^t ds\ G(k,t,s) \int_0^t dr\ G(k,t,r) k^2 \frac{d^2}{dk^2} [G(k, \mid s,r \mid) Q(k,r)]$$

understood to act on products of mean velocity gradients by integration over $r$ and $s$ and $(ij)$ denotes index interchange. In Eq. (43), $a^{(N)}$,...$f^{(N)}$ are the following geometric constants:

| | (1) | (2) | (3) | (4) | (5) | (6) |
|---|---|---|---|---|---|---|
| $105a^{(N)}$ | 27 | $-1$ | $-2$ | $\frac{19}{2}$ | $-\frac{1}{2}$ | $-1$ |
| $105b^{(N)}$ | 20 | 6 | $-2$ | 6 | 3 | $-1$ |
| $105c^{(N)}$ | $-15$ | $-15$ | $-2$ | $\frac{5}{2}$ | $-\frac{15}{2}$ | $-1$ |
| $105d^{(N)}$ | 20 | $-8$ | $-2$ | $-22$ | $-4$ | $-1$ |
| $105e^{(N)}$ | 10 | 24 | 6 | 3 | 12 | 3 |
| $105f^{(N)}$ | $-4$ | 24 | 6 | 10 | 12 | 3 |

and

$$G(k, \mid s,r \mid) = G(k,s,r) + G(k,r,s)$$

As in all derivations of this type[10,12,13,14], these constants arise from integrating even order products $k_i k_j$, .... over spheres $k$ = constant. It is easily verified that these results reduce to Maxey's[10] second order RDT calculation when $G \equiv 1$.

In extending the relaxation approximation to second order terms in the strain, $Q$ and $\partial U_i/\partial x_j$ can be taken independent of time in Eq. (43). In this case, the integrals in Eq. (44) satisfy

$$I^{(4)} = 2I^{(1)} \quad I^{(5)} = 2I^{(2)} \quad I^{(6)} = 2I^{(3)}$$

Taking traces in Eqs. (43) and (44), we find that

$$\int d\mathbf{k}\ Q_{ii}(k) = \frac{4}{15} K t^2\ \frac{\partial U_p}{\partial x_q} \frac{\partial U_p}{\partial x_q} \tag{45}$$

exactly equals the energy produced at short times by the shear given by Eq. (29). This suggests absorbing the trace of Eq. (43) in the energy equation and retaining only deviators in the stress model. Although this procedure is strictly valid only at short times, we will generalize it to arbitrary times, recognizing that this is an additional assumption. Another important property of the geometric coefficients in Eq. (43) is

$$\begin{aligned} &2(a^{(N)} + 2a^{(N+3)}) \frac{\partial U_i}{\partial x_p} \frac{\partial U_j}{\partial x_p} + (b^{(N)} + 2b^{(N+3)} + c^{(N)} + 2c^{(N+3)}) (\frac{\partial U_i}{\partial x_p} \frac{\partial U_p}{\partial x_j} + \frac{\partial U_j}{\partial x_p} \frac{\partial U_p}{\partial x_i}) \\ &+ 2(d^{(N)} + 2d^{(N+3)}) \frac{\partial U_p}{\partial x_i} \frac{\partial U_p}{\partial x_j} \\ &= (a^{(N)} + 2a^{(N+3)}) (\frac{\partial U_i}{\partial x_p} S_{pj} + \frac{\partial U_j}{\partial x_p} S_{pi}) + (d^{(N)} + 2d^{(N+3)}) (\frac{\partial U_p}{\partial x_i} S_{pj} + \frac{\partial U_p}{\partial x_j} S_{pi}) \end{aligned} \tag{46}$$

The importance of this condition, which states that the stress vanishes when the strain rate tensor $S_{ij}$ vanishes, was first clarified by Speziale.[12,13,14] It permits a consolidation of the perturbation expansion by replacing time integrals of $S$ by stresses; compare in this respect the procedures of Ref. 26 and 27. These relations have recently been verified in Yoshizawa's theory.[28]

The second order analogs of the quantities $t^{(i)}$ of Eqs. (34), (35) are

$$
\begin{aligned}
s^{(i,j)} &= \int_0^\infty dk \int_0^t ds\ (k\eta')^i (t-s)^i G(k,t,s)^2 \int_0^s dr\ (k\eta')^j (s-r)^j G(k,s,r)^2 E(k) \\
s^{(i,j)'} &= \int_0^\infty dk \int_0^t ds\ (k\eta')^i (t-s)^i G(k,t,s)^2 \int_0^s dr\ (k\eta')^j (s-r)^j G(k,s,r)^2 \times \\
&\quad \frac{d}{dk}(kE) \qquad (47) \\
s^{(i,j)''} &= \int_0^\infty dk \int_0^t ds\ (k\eta')^i (t-s)^i G(k,t,s)^2 \int_0^s dr\ (k\eta')^j (s-r)^j G(k,s,r)^2 \frac{d^2}{dk^2}(k^2 E)
\end{aligned}
$$

where $0 \le i,j \le 2$ and $0 \le i+j \le 2$. The term $s^{(0,0)}$ is of order $t^2$ at short times. The terms $s^{(i,j)}$ with $(i,j) \ne (0,0)$ are evidently of order $t^3$ or higher; integration by parts shows that $s^{(0,0)'}$ and $s^{(0,0)''}$ are also of order at least $t^3$. Thus, short time limit of the second order quantities is determined by terms containing $s^{(0,0)}$. It is shown in Appendix I that that these terms can be written as

$$
\tau_{ij}^{(2)} = \sigma_{ij}^{(1)} - 3\sigma_{ij}^{(2)} + 12\sigma_{ij}^{(6)} \tag{48}
$$

where the quantities $\sigma^{(1)}, \sigma^{(2)}, \sigma^{(6)}$ satisfy relaxation equations

$$
\begin{aligned}
\dot\sigma_{ij}^{(1)} &= T_{ij}^{(1,0)} - C_R^{(0,0)} \Theta \sigma_{ij}^{(1)} \\
\dot\sigma_{ij}^{(2)} &= T_{ij}^{(2,0)} - C_R^{(0,0)} \Theta \sigma_{ij}^{(2)} \\
\dot\sigma_{ij}^{(6)} &= T_{ij}^{(3,0)} - C_R^{(0,0)} \Theta \sigma_{ij}^{(6)}
\end{aligned} \tag{49}
$$

The relaxation constant $C_R^{(0,0)}$ is defined by a quotient analogous to Eq. (37) given in Appendix I, and the $T^{(i,j)}$ are defined by

$$
\begin{aligned}
T_{mn}^{(ij)} = (a^{(i)} + 2a^{(i+3)}) &\{ \frac{\partial U_m}{\partial x_p} T_{pn}^{(j)} + \frac{\partial U_n}{\partial x_p} T_{pm}^{(j)} - \frac{1}{3}\delta_{ij} \frac{\partial U_q}{\partial x_p} T_{pq}^{(j)} \} \\
+ (d^{(i)} + 2d^{(i+3)}) &\{ \frac{\partial U_p}{\partial x_m} T_{pn}^{(j)} + \frac{\partial U_p}{\partial x_n} T_{pm}^{(j)} - \frac{1}{3}\delta_{ij} \frac{\partial U_q}{\partial x_p} T_{qp}^{(j)} \}
\end{aligned} \tag{50}
$$

in terms of the quantities $T^{(i)}$ defined in Eq. (27). Absorbing the $r$ integration this way is justified by the property Eq. (46) of the second order geometric constants, and taking deviators is justified by the discussion following Eq. (45).

For simple shear flow, the model defined by Eq. (48) predicts that the short time dimensionless Reynolds normal stresses deviators are

$$
b_{11} = \frac{2}{3}\frac{8}{105} S^2 t^2 \quad b_{22} = \frac{2}{3}\frac{-13}{105} S^2 t^2 \quad b_{33} = \frac{2}{3}\frac{5}{105} S^2 t^2
$$

in agreement with Maxey's RDT calculation;[10] in particular, the normal stress ratios are given at short time by Eq. (4). However, this model predicts the same normal stress ratios at long and short times. The long time ratios can be corrected by adding terms to the model which vanish to higher order at short times. As noted earlier, such terms originate with the $k$ derivatives in the shear operator Eq. (14). A complete model based on analysis of all the terms is presented in Appendix I. Since it is only remotely possible that so elaborate a model could be useful in practice, we will present a simple model with the correct long time stress ratios. Using the notation for components of the second order stress in Appendix 1, this model is

$$\tau_{ij}^{(2)} = \sigma_{ij}^{(1)} - 3\sigma_{ij}^{(2)} + 12\sigma_{ij}^{(6)} - 4\sigma_{ij}^{(8)} \tag{51}$$

where the stress components satisfy relaxation equations Eq. (49) above and

$$\dot{\sigma}_{ij}^{(8)} = T_{ij}^{(3,1)} - C_R^{(0,1)}\Theta\sigma_{ij}^{(8)} \tag{52}$$

Adding the term $-4\sigma^{(8)}$ corrects the long time normal stress ratios. Here, the coefficient $-4$ has been adjusted so that assuming

$$C_R^{(0,0)} = C_R^{(0,1)} = C_R^*$$

the long time limit is

$$b_{11} = \frac{1}{C_R^* C_R^0}\frac{16}{105}(\frac{SK}{\varepsilon})^2 \quad b_{22} = \frac{1}{C_R^* C_R^0}\frac{-12}{105}(\frac{SK}{\varepsilon})^2 \quad b_{33} = \frac{1}{C_R^* C_R^0}\frac{-4}{105}(\frac{SK}{\varepsilon})^2$$

Setting the phenomenological constant

$$C_R^* = 2.00$$

gives $b_{11} = .19$ in equilibrium flows for which $SK/\varepsilon = 3.3$. As the normal stresses are also in the ratio Eq. (5), the long time limit is in good agreement with experimental data.[1,2]

The introduction of different relaxation constants $C_R^{(i)}$ and $C^{(ij)}{}_R$ for the shear and normal stresses has been advocated previously by Weinstock.[19] Many of the phenomena discussed in Ref. 19 could be described by models of this type.

## V. Conclusions

Leslie's perturbative treatment of the DIA equations for shear flow has led to a model of second order effects with the following properties:

1. The model consists of linear relaxation equations. This attribute should be compared with models nonlinear in the stresses recently criticized by Speziale:[1,2] such nonlinearity can be problematical in connection with rapid distortion limits.
2. The model correctly accounts for the change in Reynolds normal stress ratio from short to long times. This property is interesting from a theoretical standpoint despite the limited practical importance of the short time normal stress ratios.

3. In this model, the shear stress determines the normal stresses, which occur as effects of higher order. This property contrasts with stress transport models like LRR in which the normal stresses influence the shear stresses.

The limitation of the model to small strain rates and to strain rate histories which are not extremely complex must be noted. This model can be applied to predict normal stresses in oscillating flows. This problem has been successfully treated using the RDT based model by Mankbadi and Brereton,[16,17] however, as noted previously their model slightly modifies the short time ratios.

## Appendix I. Derivation of second order relaxation model

The second order calculation requires the quantity

$$t^{(2)} = \int_0^\infty dk \int_0^t ds\ (k\eta')^2 (t-s)^2 G(k,t,s)^2 E$$

analogous to $t^{(0)}$ and $t^{(1)}$ in Eqs. (34), (35). The relaxation model

$$\dot{t}^{(2)} = 2C_S^{(2)}\Theta t^{(1)} - C_R^{(2)}\Theta t^{(2)}$$

is derived like the model for $t^{(1)}$; the result is

$$C_S^{(2)}\Theta = \frac{\int_0^\infty dk\ (k\eta')^2\theta_1 E}{\int_0^\infty dk\ k\eta'\theta_1 E} \quad C_R^{(2)}\Theta = \frac{\int_0^\infty dk\ 2\eta\ (k\eta')^2\ \theta_2 E}{\int_0^\infty dk\ (k\eta')^2\theta_2 E} \tag{53}$$

The definitions Eq. (50) of $T^{(ij)}$ now apply when $j = 2$.

Appropriate relaxation approximations for the time evolution of the $s^{(ij)}$ Eq. (47) are

$$\begin{aligned}
\dot{s}^{(0,0)} &= t^{(0)} - C_R^{(0,0)}\Theta s^{(0,0)} \\
\dot{s}^{(1,0)} &= C_S^{(1,0)}\Theta s^{(0,0)} - C_R^{(1,0)}\Theta s^{(1,0)} \\
\dot{s}^{(0,1)} &= t^{(1)} - C_R^{(0,1)}\Theta s^{(0,1)} \\
\dot{s}^{(2,0)} &= 2C_S^{(2,0)}\Theta s^{(1,0)} - C_R^{(2,0)}\Theta s^{(2,0)} \\
\dot{s}^{(1,1)} &= C_S^{(1,1)}\Theta s^{(0,1)} - C_R^{(1,1)}\Theta s^{(1,1)} \\
\dot{s}^{(0,2)} &= t^{(2)} - C_R^{(0,2)}\Theta s^{(0,2)}
\end{aligned}$$

where the relaxation constants are determined by

$$\begin{aligned}
C_R^{(0,0)}\Theta &= \frac{\int_0^\infty dk\ 2\eta\theta_1 E}{\int_0^\infty dk\ \theta_1 E} \\
C_S^{(1,0)}\Theta &= \frac{\int_0^\infty dk\ k\eta'\theta_1 E}{\int_0^\infty dk\ \theta_1 E} \quad C_R^{(1,0)}\Theta = \frac{\int_0^\infty dk\ 2\eta k\eta'\theta_2 E}{\int_0^\infty dk\ k\eta'\theta_2 E} \\
C_S^{(2,0)}\Theta &= \frac{\int_0^\infty dk\ (k\eta')^2\theta_2 E}{\int_0^\infty dk\ k\eta'\theta_2 E} \quad C_R^{(2,0)}\Theta = \frac{\int_0^\infty dk\ 2\eta(k\eta')^2\theta_3 E}{\int_0^\infty dk\ (k\eta')^2\theta_3 E} \\
C_S^{(1,1)}\Theta &= \frac{\int_0^\infty dk(k\eta')^2\theta_2 E}{\int_0^\infty dk\ k\eta'\theta_2 E} \quad C_R^{(1,1)}\Theta = \frac{\int_0^\infty dk\ 2\eta(k\eta')^2\theta_3 E}{\int_0^\infty dk\ (k\eta')^2\theta_3 E}
\end{aligned} \tag{54}$$

The remaining relaxation constants satisfy

$$C_R^{(0,1)} = C_R^{(0,1)} \quad C_R^{(0,2)} = C_R^{(1,1)} = C_R^{(2,0)}$$

The second order part of the stress is the sum of contributions from $I^{(N)} + 2I^{(N+3)}$ for $N = 1, 2, 3$. The contribution from $I^{(1)} + 2I^{(4)}$ is

$$\sigma^{(1)} = \int_0^\infty dk \int_0^t \; ds\; G(k,t,s)^2 \int_0^s \; dr\; G(k,s,r)^2 E(k)\{(a^{(1)} + 2a^{(4)})(\nabla US + S\nabla U^T) \\ + (d^{(1)} + 2d^{(4)}(\nabla U^T S + S\nabla U)\}_D$$

where matrix notation is used and the subscript $D$ denotes deviatoric part. The relaxation approximation is

$$\dot{\sigma}^{(1)} = T^{(1,0)} - C_R^{(0,0)}\sigma^{(1)} \tag{55}$$

The contribution from $I^{(2)} + 2I^{(5)}$ can be written

$$\int d\mathbf{k} \int_0^t \; ds\; G(k,t,s) \int_0^s \; dr\; G(k,s,r) k \frac{d}{dk}[G(k,t,r)Q(k)] \times \\ \{(a^{(2)} + a^{(5)})(\nabla US + S\nabla U^T) + (d^{(2)} + d^{(5)})(\nabla U^T S + S\nabla U)\}_D \\ = -3\sigma^{(2)} - \sigma^{(3)} - \sigma^{(4)} + \sigma^{(5)}$$

where

$$\begin{aligned} \sigma^{(2)} &= s^{(0,0)}\{(a^{(2)} + 2a^{(5)})[\nabla US + S\nabla U^T] + (d^{(2)} + 2d^{(5)})[\nabla U^T S + S\nabla U]\}_D \\ \sigma^{(3)} &= s^{(1,0)}\{(a^{(2)} + 2a^{(5)})[\nabla US + S\nabla U^T] + (d^{(2)} + 2d^{(5)})[\nabla U^T S + S\nabla U]\}_D \\ \sigma^{(4)} &= s^{(0,1)}\{(a^{(2)} + 2a^{(5)})[\nabla US + S\nabla U^T] + (d^{(2)} + 2d^{(5)})[\nabla U^T S + S\nabla U]\}_D \\ \sigma^{(5)} &= s^{(0,0)'}\{(a^{(2)} + 2a^{(5)})[\nabla US + S\nabla U^T] + (d^{(2)} + 2d^{(5)})[\nabla U^T S + S\nabla U]\}_D \end{aligned}$$

and

$$s^{(0,0)'} = \int_0^\infty dk\; G(k,t,s)^2 \int G(k,s,r)^2 \frac{d}{dk}(kE)$$

In the relaxation approximation,

$$\begin{aligned} \dot{\sigma}^{(2)} &= T^{(2,0)} - C_R^{(0,0)}\Theta\sigma^{(2)} \\ \dot{\sigma}^{(3)} &= C_S^{(1,0)}\Theta T^{(2,0)} - C_R^{(1,0)}\Theta\sigma^{(3)} \\ \dot{\sigma}^{(4)} &= T^{(2,1)} - C_R^{(0,1)}\Theta\sigma^{(4)} \\ \dot{\sigma}^{(5)} &= 2T^{(2,1)} - C_R^{(0,0)'}\Theta\sigma^{(5)} \end{aligned} \tag{56}$$

where

$$C_R^{(0,0)'}\Theta = \frac{\int_0^\infty dk\; 2\eta\theta_1(kE)'}{\int_0^\infty dk\; \theta_1(kE)'} \tag{57}$$

Integral $I^{(3)}$ can be written

$$I^{(3)} = I^{(3,1)} + 2I^{(3,2)} + I^{(3,3)}$$

where

$$\begin{aligned}
I^{(3,1)} &= \int_0^t \ ds\ G(k,t,s) \int_0^s \ dr\ G(k,s,r) k^2 [\frac{d^2}{dk^2} G(k,t,r)] k^2 Q \\
I^{(3,2)} &= \int_0^t \ ds\ G(k,t,s) \int_0^s \ dr\ G(k,s,r) [k \frac{d}{dk} G(k,t,r)][k^3 \frac{dQ}{dk}] \\
I^{(3,3)} &= \int_0^t \ ds\ G(k,t,s)^2 \int_0^s \ dr\ G(k,s,r)^2 k^4 \frac{d^2 Q}{dk^2}
\end{aligned}$$

The term proportional to $I^{(3,1)}$ can be written as

$$\begin{aligned}
&\int d\mathbf{k}\ I^{(3,1)} \big\{ (a^{(3)} + a^{(6)})(\nabla U S + S \nabla U^T) + (d^{(3)} + d^{(6)})(\nabla U^T S + S \nabla U) \big\}_D \\
&\quad = -C^{(7)} \sigma^{(7)} - C^{(8)} \sigma^{(8)} + \sigma^{(9)} + 2\sigma^{(10)} + \sigma^{(11)}
\end{aligned}$$

where

$$\begin{aligned}
\sigma^{(6)} &= s^{(0,0)} \big\{ (a^{(3)} + 2a^{(6)})[\nabla U S + S \nabla U^T] + (d^{(3)} + 2d^{(6)})[\nabla U^T S + S \nabla U] \big\}_D \\
\sigma^{(7)} &= s^{(1,0)} \big\{ (a^{(3)} + 2a^{(6)})[\nabla U S + S \nabla U^T] + (d^{(3)} + 2d^{(6)})[\nabla U^T S + S \nabla U] \big\}_D \\
\sigma^{(8)} &= s^{(0,1)} \big\{ (a^{(3)} + 2a^{(6)})[\nabla U S + S \nabla U^T] + (d^{(3)} + 2d^{(6)})[\nabla U^T S + S \nabla U] \big\}_D \\
\sigma^{(9)} &= s^{(2,0)} \big\{ (a^{(3)} + 2a^{(6)})[\nabla U S + S \nabla U^T] + (d^{(3)} + 2d^{(6)})[\nabla U^T S + S \nabla U] \big\}_D \\
\sigma^{(10)} &= s^{(1,1)} \big\{ (a^{(3)} + 2a^{(6)})[\nabla U S + S \nabla U^T] + (d^{(3)} + 2d^{(6)})[\nabla U^T S + S \nabla U] \big\}_D \\
\sigma^{(11)} &= s^{(0,2)} \big\{ (a^{(3)} + 2a^{(6)})[\nabla U S + S \nabla U^T] + (d^{(3)} + 2d^{(6)})[\nabla U^T S + S \nabla U] \big\}_D
\end{aligned}$$

and the quantities $C^{(7)}$ and $C^{(8)}$ are

$$C^{(7)} = C^{(8)} = \frac{\int_0^\infty dk\ k^2 \eta''\ \theta_2 E}{\int_0^\infty dk\ k \eta' \theta_2 E}$$

Relaxation approximations for the time evolution of these terms are

$$\begin{aligned}
\dot{\sigma}^{(6)} &= T^{(3,0)} - C_R^{(0,0)} \Theta \sigma^{(6)} \\
\dot{\sigma}^{(7)} &= C_S^{(1,0)} \Theta \sigma^{(6)} - C_R^{(1,0)} \Theta \sigma^{(7)} \\
\dot{\sigma}^{(8)} &= T^{(3,1)} - C_R^{(0,1)} \Theta \sigma^{(8)} \\
\dot{\sigma}^{(9)} &= 2C^{(2,0)} \Theta \sigma^{(7)} - C_R^{(2,0)} \Theta \sigma^{(9)} \\
\dot{\sigma}^{(10)} &= \sigma^{(8)} - C_R^{(1,1)} \Theta \sigma^{(10)} \\
\dot{\sigma}^{(11)} &= T^{(3,2)} - C_R^{(0,2)} \Theta \sigma^{(11)}
\end{aligned} \tag{58}$$

To evaluate the $I^{(3,2)}$ term, write

$$k^3 \frac{dQ}{dk} = \frac{d}{dk}(k^3 Q) - 3k^2 Q$$

then

$$\int d\mathbf{k}\ I^{(3,2)}\left\{(a^{(3)} + a^{(6)})(\nabla US + S\nabla U^T) + (d^{(3)} + d^{(6)})(\nabla U^T S + S\nabla U)\right\}_D$$
$$= 3\sigma^{(7)} + 3\sigma^{(8)} - \sigma^{(12)} - \sigma^{(13)}$$

where

$$\sigma^{(12)} = s^{(1,0)} \frac{d}{dk}(kE)$$
$$\sigma^{(13)} = s^{(0,1)} \frac{d}{dk}(kE)$$

and the relaxation approximations are

$$\dot{\sigma}^{(12)} = -C_S^{(0,0)'}\Theta\sigma^{(6)} + 2C_S^{(1,0)'}\Theta\sigma^{(7)} + C_S^{(0,1)'}\Theta\sigma^{(8)} - C_R^{(1,1)'}\Theta\sigma^{(12)} \tag{59}$$

where the modified transport coefficients are

$$C_S^{(0,0)'}\Theta = \frac{\int_0^\infty dk\ k(k\eta')'\theta_1 E}{\int_0^\infty dk\ \theta_1 E} \quad C_S^{(1,0)'}\Theta = \frac{\int_0^\infty dk\ (k\eta')^2 \frac{1}{2}\theta_2 E}{\int k\eta' \frac{1}{2}\theta_2 E}$$
$$C_S^{(0,1)'}\Theta = \frac{\int_0^\infty dk\ (k\eta')^2 \frac{1}{2}\theta_2 E}{\int_0^\infty dk\ k\eta' \frac{1}{2}\theta_2 E} \quad C_R^{(1,1)'}\Theta = \frac{\int_0^\infty dk\ 2\eta k\eta' \frac{1}{2}\theta_2 (kE)'}{\int_0^\infty dk\ k\eta' \frac{1}{2}\theta_2 (kE)'} \tag{60}$$

and

$$\dot{\sigma}^{(13)} = T^{(3,1)} + 2T^{(3,2)} - C_R^{(0,1)'}\Theta\sigma^{(13)} \tag{61}$$

where

$$C_R^{(0,1)'}\Theta = \frac{\int_0^\infty dk\ 2\eta k\eta' \frac{1}{2}\theta_2 (kE)'}{\int_0^\infty dk\ k\eta' \frac{1}{2}\theta_2 (kE)'} \tag{62}$$

To evaluate the contribution from $I^{(3,3)}$, write

$$k^4 Q'' = 12k^2 Q - 8(k^3 Q)' + (k^4 Q)''$$

Then

$$\int d\mathbf{k}\ I^{(3,3)}\left\{(a^{(3)} + a^{(6)})(\nabla US + S\nabla U^T) + (d^{(3)} + d^{(6)})(\nabla U^T S + S\nabla U)\right\}_D$$
$$= 12\sigma^{(6)} + \sigma^{(14)} - 8\sigma^{(15)}$$

where

$$\sigma^{(14)} = s^{(0,0)} \frac{d}{dk}(k^3 Q)$$

has the relaxation approximation

$$\dot{\sigma}^{(14)} = 2T^{(3,1)} - C_R^{(0,0)'}\Theta\sigma^{(14)} \tag{63}$$

and

$$\sigma^{(15)} = s^{(0,0)}\frac{d^2}{dk^2}k^4 Q$$

has the relaxation approximation

$$\dot{\sigma}^{(15)} = -C_S^{(0,0)''}\Theta T^{(3,1)} + T^{(3,2)} - C_R^{(0,0)''}\Theta\sigma^{(15)} \tag{64}$$

with

$$C_S^{(0,0)''}\Theta = \frac{\int_0^\infty dk\ k^2\eta''\theta_1 E}{\int_0^\infty dk\ \theta_1 E} \quad C_R^{(0,0)''}\Theta = \frac{\int_0^\infty dk\ 2\eta\theta_1(k^2E)''}{\int_0^\infty dk\ \theta_1(k^2E)''} \tag{65}$$

To summarize, the second order model is

$$\tau^{(2)} = \sigma^{(1)} - 3\sigma^{(2)} - \sigma^{(3)} - \sigma^{(4)} + \sigma^{(5)} + 12\sigma^{(6)} + (3 - C^{(7)})\sigma^{(7)} + (3 - C^{(8)})\sigma^{(8)} + \sigma^{(9)}$$
$$+ 2\sigma^{(10)} + \sigma^{(11)} - \sigma^{(12)} - \sigma^{(13)} + \sigma^{(14)} - 8\sigma^{(15)}$$

where the $\sigma^{(p)}$ satisfy the system of relaxation equations Eqs. (55), (56), (58), (61), (63), (64) and the constants are defined by Eqs. (53), (54), (57), (60), (62), (65).

## Appendix II. Markovianized DIA

Deriving useful conclusions from Leslie's theory requires analytical expressions for the descriptors $G^{(0)}$ and $Q^{(0)}$ of the isotropic background field. Suitable expressions will be obtained from a Markovianized DIA derived by evaluating the time dependence in the distant interaction limit. Let the real function $H(\xi), 0 \le \xi < \infty$ satisfy

$$H(0) = 1, \quad H(\xi) < 1 \text{ for } \xi > 0, \quad \int_0^\infty H(\xi)d\xi \ < \ \infty \tag{66}$$

Then standard properties of delta functions imply

$$\lambda H(\lambda(t-s)) \ \sim \ \delta(t-s) \text{ for } \lambda \to \infty \tag{67}$$

Evaluate Eq. (10) in the distant interaction approximation in which $k \to 0, \quad p, q \to \infty$. Then

$$\begin{aligned}\eta^{(0)}(\mathbf{k},t,s) &= \int^*_{\mathbf{k}=\mathbf{p}+\mathbf{q}} d\mathbf{p}d\mathbf{q}\ B(\mathbf{k},\mathbf{p},\mathbf{q})G^{(0)}(p,t,s)Q^{(0)}(q,t,s) \\ &\sim \int^* d\mathbf{p}\ \{k_m\frac{\partial B}{\partial q_m}(\mathbf{k},\mathbf{p},\mathbf{p})G^{(0)}(p,t,s)Q^{(0)}(p,t,s) \\ &- B(\mathbf{k},\mathbf{p},\mathbf{p})G^{(0)}(p,t,s)k_m p_m p^{-1}\frac{dQ^{(0)}}{dp}(p,t,s)\}\end{aligned}$$

where $B(\mathbf{k},\mathbf{p},\mathbf{q})$ denotes the product of projection operators in Eq. (10) and $\int^*$ denotes infrared regularization by restriction of the region of integration to $(\mathbf{p},\mathbf{q})$ pairs satisfying $p,q \geq k$.[29] Assuming time stationary similarity forms $G^{(0)}(p,t,s) = G^{(0)}(p^r(t-s))$ and $Q^{(0)}(p,t,s) = R^{(0)}(p^r(t-s))Q^{(0)}(p)$, the properties Eq. (66) of $H$ may reasonably be postulated of the product $G^{(0)}R^{(0)}$. Therefore, Eq. (67) implies that in this limit the damping is Markovian

$$\eta^{(0)}(\mathbf{k},t,s) \;=\; \delta(t-s)\eta^{(0)}(\mathbf{k})$$

and Eq. (7) implies that the Green's function is exponential,

$$G^{(0)}(k,t,s) = \exp\,[(s-t)\eta^{(0)}(\mathbf{k})] \quad \text{for } t \geq s \tag{68}$$

Likewise evaluating the force correlation Eqs. (11), (12) in the distant interaction limit implies that the forcing is white noise in time:

$$< f_i^{(0)}(\mathbf{k},t) f_j^{(0)}(\mathbf{k}',s) > = \;\delta(t-s)\delta(\mathbf{k}+\mathbf{k}')F_{ij}^{(0)}(\mathbf{k}) \tag{69}$$

Computing the correlation function from the relation

$$\begin{aligned} Q^{(0)}(\mathbf{k},t,s)\delta(\mathbf{k}+\mathbf{k}') \;&=\; \int_0^t dr_1\; G^{(0)}(k,t,r_1) \int_0^t dr_2\; G^{(0)}(k',t,r_2) \times \\ &\quad < f^{(0)}(\mathbf{k},r_1) f^{(0)}(\mathbf{k}',r_2) > \end{aligned}$$

using Eqs. (68), (69) shows that the fluctuation dissipation relation

$$Q^{(0)}(k,t,s) = Q^{(0)}(k)\big[G^{(0)}(k,t,s) + G^{(0)}(k,s,t)\big] \tag{70}$$

expressing the time dependence of the correlation functions in terms of the response function is also valid in this limit.

## REFERENCES


1. C. G. Speziale, "Analytical methods for the development of Reynolds stress closures in turbulence," Ann. Rev. Fluid Mech. **23** 107 (1991).
2. C. G. Speziale, "A review of Reynolds stress models for turbulent shear flows," 20th Symposium on Naval Hydrodynamics (1994).
3. D. C. Leslie, *Modern Developments in the Theory of Turbulence*, Oxford Press (1972).
4. R. H. Kraichnan, "Direct interaction approximation for shear and thermally driven turbulence," Phys. Fluids, **7**, 1048 (1964).
5. A. Yoshizawa, "Statistical theoretical turbulence modeling in hydrodynamic flows," University of Tokyo preprint (1995).
6. V. Yakhot, S. Thangam, T. Gatski, S. A. Orszag, and C. G. Speziale, "Development of turbulence models for shear turbulence by a double expansion technique," Phys. Fluids A, **4** 1510 (1992).
7. L. M. Smith and V. Yakhot, "Short-time and long-time behavior of eddy viscosity models," Theor. Comput. Fluid Dyn. **4**, 197 (1993).

8. S. Crow, "Viscoelastic properties of fine grained turbulence," J. Fluid Mech. **41**, 81 (1968).
9. K. Hanjalic and B. E. Launder, "A Reynolds stress model of turbulence and its application to thin shear layers," J. Fluid Mech. **52**, 609 (1972).
10. M. R. Maxey, "Distortion of turbulence by uniform shear," J. Fluid Mech. **124**, 261 (1982).
11. S. Pope, "A more general effective-viscosity hypothesis," J. Fluid Mech. **72**, 331 (1975).
12. A. Yoshizawa, "Statistical analysis of the deviation of the Reynolds stress from its eddy viscosity representation," Phys. Fluids, **27**, 1377 (1984).
13. C. G. Speziale, "On nonlinear K-1 and K-e models of turbulence," J. Fluid Mech. **178**, 459 (1987).
14. R. Rubinstein and J. M. Barton, "Nonlinear Reynolds stress models and the renormalization group," Phys. Fluids A, **2**, 1472 (1990).
15. B. E. Launder, G. J. Reece, and W. Rodi, "Progress in the development of a Reynolds stress turbulence closure," J. Fluid Mech. **68**, 537 (1975).
16. R. R. Mankbadi and J. T. C. Liu, "Near-wall response in turbulent shear flows subjected to imposed unsteadiness," J. Fluid Mech. **238**, 55 (1992).
17. G. J. Brereton and R. R. Mankbadi, "A rapid distortion theory turbulence model for developed unsteady wall bounded flow," NASA Technical Memorandum TM-106249 (1993).
18. J. O. Hinze, *Turbulence*, McGraw-Hill (1975).
19. J. Weinstock, "Theory of pressure-strain rate correlation for Reynolds stress turbulence closures," J. Fluid Mech. **105**, 369 (1981).
20. R. H. Kraichnan, "Eddy viscosity in two and three dimensions," J. Atmos. Sci. **33**, 1521 (1976).
21. C. E. Leith, "Atmospheric predictability and two dimensional turbulence," J. Atmos. Sci. **28**, 145 (1971).
22. H. Mori, "Transport, collective motion and Brownian motion," Prog. Theor. Phys. **33**, 423 (1965).
23. J. A. Domaradzki and S. A. Orszag, "Numerical solutions of the direct interaction equations for anisotropic turbulence," J. Sci. Comp. **2**, 227 (1987).
24. S. L. Woodruff, "On the theoretical development of models for turbulent flows with unsteady statistics," Brown University preprint (1995).
25. R. Kubo, "Statistical mechanical theory of irreversible processes," J. Phys. Soc. Japan **12**, 570 (1957).
26. A. Yoshizawa, "Bridging between eddy viscosity type and second order turbulence models through a two scale turbulence theory," Phys. Rev. E **48**, 273 (1993).
27. R. Rubinstein and J. M. Barton, "Renormalization group analysis of the Reynolds stress transport equation," Phys. Fluids A **4**, 1759 (1992).
28. M. Okamoto, "Theoretical investigation of an eddy viscosity type representation of the Reynolds stress," J. Phys. Soc. Japan **63** 2102 (1994).
29. R. H. Kraichnan, "Kolmogorov's hypotheses and Eulerian turbulence theory," Phys. Fluids **7**, 1723 (1964).